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THE WALL PRESSURE DISTRIBUTION IN A CHOKED TUNNEL

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AIRCRAFT LABORATORY

DECEMBER 1953

WRIGHT AIR DEVELOPMENT CENTER

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WADC TECHNICAL REPORT 53-509

THE WALL PRESSURE DISTRIBUTION IN A CHOKED TUNNEL

Gottfried Guderley

Aircraft Laboratory

December 1953

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Wright Air Development Center
Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

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FOREWORD

This report was prepared by the Wind Tunnel Branch, Aircraft Laboratory, Directorate of Laboratories, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. The project was administered under Research and Development Order Number 458-429, (UNCLASSIFIED), "Two-Dimensional and Axially Symmetric Transonic Flow," with Gottfried Guderley acting as project engineer.

ABSTRACT

The wall pressure distribution in a choked tunnel is computed, following a method outlined previously. By the calculus of residues the solution is brought into such a form that its structure is displayed more clearly.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

for Carl Reichert

DANIEL D. McKEE
Colonel, USAF
Chief, Aircraft Laboratory
Directorate of Laboratories

TABLE OF CONTENTS

LIST OF SYMBOLS

x, y	Cartesian coordinates
ψ	stream function
w, θ	absolute value and direction of the velocity vector
w^*	sonic velocity
γ	ratio of the specific heats
η	independent variable defined in Equation (1)
ρ, ζ	independent variables defined in Equation (3)
$g(\rho)$	function occurring in Equation (5)
$G(\zeta, \lambda)$ $G^a(\zeta, \lambda)$ $G^s(\zeta, \lambda)$	functions defined in the paragraph subsequent to Equation (7)
λ_h	eigen values
$\psi_{I,h}$ $\psi_{II,h}$, $\psi_{III,h}$, $\psi_{IV,h}$	particular solutions defined in Equation (9)
h	integral number
c_2	constant used in the definition of eigen value problem
ρ_o	special value of ρ (value of ρ for point A)
η_o	special value of η (value of η for point A)
$f(\zeta)$	arbitrary function
C_h	normalization constant (Equation (11))
τ	variable of integration
ν	quantity connected to the negative eigen values (Equation (16))
ε	quantity connected to c_2 (Equation (17))
$A(\nu)$, $\sigma(\nu)$	functions defined by Equation (18)
$H(\zeta)$	function defined in Equation (23)

LIST OF SYMBOLS (Continued)

c_3	constant occurring in a formula for H
a_h, b_h	coefficients of infinite series for the solutions (Equations (5) and (27))
$\alpha(\nu)$, $\beta(\nu)$	functions occurring in Fourier integrals
ψ_h, $\tilde{\psi}_h$	particular solution defined in Equation (26)
H	height (width) of the wind tunnel
J	integral defined in Equation (31)
u	variable of integration connected to ν
M	Mach number
M_{chok}	choking Mach number
p	pressure
p*	pressure at sonic speed
p_{chok}	pressure at the choking Mach number

INTRODUCTION

For experiments on the wall influence in transonic flow the wall pressure distribution in a choked tunnel has some interest. It is determined in the present report under the assumption that the wind tunnel is very wide in comparison to the model dimensions.

The paper is closely related to an investigation by B. Marschner (Reference 1) on the influence of the tunnel wall on the pressure distribution over the body. Indeed his analysis contains in principle all ideas necessary for the determination of the wall pressure distribution. However, the solution, thus obtained, would appear in a form which is not yet suited for the numerical evaluation and does not put into evidence its essential properties. The main task of the present report is a transformation of the original result into a more suitable form by means of the calculus of residues.

An aerodynamicist will consider as essential in this paper the pressure distribution over the wind-tunnel wall. But the mathematical part is also quite enjoyable because of the spectacular simplifications which are achieved.

Since the problem is explained in full detail in the paper of Marschner, and since it follows the procedure outlined in Reference 2, the general explanations can be kept rather brief.

SECTION I

THE BOUNDARY VALUE PROBLEM

Figure 1 shows the flow in the physical plane. If the tunnel is choked, a sonic line (OC) will extend from the body to the wind-tunnel wall. Mach waves which start from the supersonic contour of the body will either end at the sonic line or at the wind-tunnel wall. The one Mach wave which separates these two classes is called the limiting Mach wave, since it gives the downstream limit of the part of the supersonic flow field which influences the subsonic flow (line OD).

The hodograph picture is shown in Figure 2. To fix the ideas the body has been assumed to be a wedge. Corresponding points in the physical plane and in the hodograph plane have the same notation. The surface of the wedge maps into the line BC, the wind-tunnel wall into the line AO (with different values of the stream function on the upper and lower side of this line, since they correspond to the upper and lower walls of the tunnel). At point A a singularity arises since it corresponds to the flow in the tunnel at a large distance upstream and thus all streamlines originate at this point. AB corresponds to the line of symmetry of the wind tunnel; along this line the value of the stream function is zero.

The sharp corner which occurs at C in the physical plane maps into the characteristic CD, which will be referred to as the shoulder characteristic.

In the transonic approximation one usually introduces instead of the coordinate w (w absolute value of the velocity vector) another variable η, defined by

$$w = w^* \left(1 + (\gamma + 1)^{-\frac{1}{3}} \eta\right) \tag{1}$$

where w^* is the sonic velocity and γ is the ratio of the specific heats. The other independent variable is θ, the angle of the velocity vector with the horizontal axis. Actually, a limiting process is carried out in which it is assumed that the body becomes very slender and the Mach number very close to one. By this limiting process the value of η for the stagnation point moves to $-\infty$. The boundary value problem in the η, θ-plane, is shown in Figure 3.

This problem will be treated with a method described in Reference 2. The approach chosen has the advantage that the solutions appear in a particularly simple form in the limiting case, where the choking Mach

number is close to one; or, in other words, if the wind-tunnel dimensions are large in comparison to the model.

SECTION II

PARTICULAR SOLUTIONS

In the transonic approximation the equation for the stream function ψ assumes the form

$$\psi_{\eta\eta} - \eta\,\psi_{\theta\theta} = 0 \tag{2}$$

Introducing new coordinates

$$\xi = \frac{\eta}{\left(\frac{3}{2}\theta\right)^{2/3}} \tag{3a}$$

$$\rho = -\eta^3 + \left(\tfrac{3}{2}\theta\right)^2 \tag{3b}$$

one obtains as the equation for the stream function

$$\frac{(1-\xi^3)^2}{\xi}\cdot\psi_{\xi\xi} - \frac{5}{2}\xi(1-\xi^3)\psi_\xi + \frac{1}{16}\psi = 9\rho^2\cdot\psi_{\rho\rho} + \frac{21}{2}\rho\,\psi_\rho + \frac{1}{16}\psi \tag{4}$$

Particular solutions of the form

$$\psi = g(\rho)\cdot G(\xi) \tag{5}$$

have been studied in Reference 2. They yield the differential equation for G

$$\frac{(1-\xi^3)^2}{\xi}\cdot G'' - \frac{5}{2}\xi(1-\xi^3)G' + \left(-\lambda + \frac{1}{16}\right)G = 0 \tag{6}$$

which can be written as

$$\frac{d}{d\zeta}\left\{(1-\zeta^3)^{5/6}\cdot G'\right\} + \left(-\lambda+\frac{1}{16}\right)\frac{\zeta}{(1-\zeta^3)^{7/6}}\cdot G = 0 \qquad (7)$$

The function $g(\rho)$ is expressed by

$$g(\rho) = \rho^{-\frac{1}{12}\pm\frac{1}{3}\sqrt{\lambda}}$$

The function G depends upon ζ and λ ; therefore, the notation $G(\zeta, \lambda)$ will be used occasionally. The arising particular solutions for ψ will be either symmetric or antisymmetric with respect to the axis $\theta = 0$; therefore superscripts s, symmetric, and a, antisymmetric, will be used.

Let c_2 be a constant whose choice will be discussed later. If along the lines $\zeta = -\infty$ and $\zeta = c_2$ the boundary condition $\psi = 0$, and consequently $G = 0$, is prescribed, the functions G and the parameters λ are determined by an eigen value problem. It deviates from classical eigen value problems only by the fact that the coefficient of G in Equation (6) changes its sign, namely, at $\zeta = 0$, i.e., with the transition from the subsonic to the supersonic range. One obtains an infinite number of positive and an infinite number of negative eigen values and a corresponding system of eigen functions G. The eigen values may be arranged according to their magnitude and denoted by

$$\ldots \lambda_{-h} \ldots \lambda_{-3} \quad \lambda_{-2} \quad \lambda_{-1} \quad \lambda_1 \quad \lambda_2 \quad \lambda_3 \ldots \lambda_h \ldots$$

where the negative subscripts refer to negative eigen values and the positive subscripts to positive eigen values. The eigen functions G may be denoted accordingly.

The system of eigen functions can be shown to be complete. One has the following relations of orthogonality

$$\int_{-\infty}^{c_2} \frac{\zeta}{(1-\zeta^3)^{7/6}}\, G_h \cdot G_k \, d\zeta = 0 \qquad \text{for } h \neq k \qquad (8)$$

The particular solutions for ψ which can be formed with these expressions G will be denoted in the following manner:

$$\psi_{I h} = \rho^{-\frac{1}{12} + \frac{1}{3}\sqrt{\lambda_h}} \cdot G_h \tag{9a}$$

$$\psi_{II h} = \rho^{-\frac{1}{12} - \frac{1}{3}\sqrt{\lambda_h}} \cdot G_h \tag{9b}$$

$$\psi_{III h} = \rho^{-\frac{1}{12}} G_{-h}(\zeta) \cdot \cos\left(\frac{1}{3}\sqrt{|\lambda_{-h}|}\log\frac{\rho}{\rho_0}\right) \tag{9c}$$

$$\psi_{IV h} = \rho^{-\frac{1}{12}} \cdot G_{-h}(\zeta) \cdot \sin\left(\frac{1}{3}\sqrt{|\lambda_{-h}|}\log\frac{\rho}{\rho_0}\right) \tag{9d}$$

Here h is a positive integral number, ρ_0 is a suitable constant.

Because of the completeness of the system of eigen functions, any function ψ which fulfills for $\rho_1 < \rho < \rho_2$ (ρ_1 and ρ_2 arbitrary constants) the differential equation (4) and the boundary conditions $\psi = 0$ along $\zeta = -\infty$ and $\zeta = c_2$, can be represented in this region by a superposition of expressions (9). (See Reference 2, Appendix 1.)

SECTION III

THE CHOICE OF THE VALUE OF C_2

Since in the original boundary value problem no condition is prescribed along the limiting characteristic $\zeta = 1$, the application of these particular solutions to the present problem is not quite self-evident.

We proceed as follows: along the shoulder characteristic the boundary condition $\psi = 0$ is prescribed. We now define that $\psi \equiv 0$

in the region CDG of Figure 3. Then the differential equation for ψ is fulfilled everywhere in the region OABCGDO, even at the shoulder characteristic CD. (That along this line a jump of the first derivative of ψ occurs, is not in contradiction to the fulfillment of the differential equation.)

In order to prepare the use of the eigen functions, we draw a line $\zeta = c_2$ $(0 < c_2 < 1)$ and prescribe (in contradiction with the requirements of the boundary value problem), that $\psi = 0$ along $\zeta = c_2$. It has been made plausible in Reference 2 that in the limiting case, $c_2 \to 1$, the solution of the boundary value problem with the line $\zeta = c_2$ as additional boundary, tends to the solution of the original problem. For a purely supersonic problem this state of affairs is quite obvious (see Reference 2); the reasoning for the mixed subsonic supersonic case is less direct. For the present problem this is unessential, since the final form of the solution will justify the procedure. Actually, the present example may serve as an illustration for the limiting process $c_2 \to 1$.

Perhaps some remarks regarding the physical nature of this limiting process will be welcome. It seems as if, by the limiting process that $c_2 \to 1$, the boundary condition $\psi = 0$ is imposed not only along the shoulder characteristic CD but also along the limiting characteristic DO. This is in contradiction with the requirements of Tricomi's existence theorem where no condition can be imposed along one of the characteristics which form the boundary of the supersonic region. On the other hand the boundary condition has a physical significance. Let us assume that the flow field in a closed tunnel has been found up to the limiting Mach wave. Now we construct the rear part of the body in such a manner that it does not reflect any Mach waves. Then the region DQOH of Figure 4 maps into the one characteristic DO of the hodograph. Thus the ψ-surface in the hodograph plane will drop along the line DO suddenly from the values which correspond to the limiting characteristic, to the value of $\psi = 0$ which corresponds to the line DH in Figure 4. This sudden drop in the hodograph is the mapping of the so-called "lost" solutions ("lost" if only such hodograph expressions are considered which are continuous in w and θ). Thus, the requirement $\psi = 0$ along the limiting characteristic has a physical interpretation, although it is meaningless for the mathematical formulation of the problem.

Also the boundary condition $\psi = 0$ along $\zeta = c_2$ can be given a physical meaning, it requires that the Meyer expansion at the shoulder is carried out only up to point K and that from there on, a certain relation between the pressure and the streamline direction is prescribed along the zero streamline. If $c_2 \to 1$ the Meyer expansion is carried out further and further, and the condition along the zero streamline comes closer and closer to the condition assumed for the rear part of the body in Figure 4. Thus the limiting process $c_2 \to 1$ appears to be justified by a physical consideration.

SECTION IV

REPRESENTATION OF AN ARBITRARY FUNCTION BY A SUPERPOSITION OF EIGEN FUNCTIONS G

An arbitrary function $f(\zeta)$ can be represented by a Fourier series in the eigen functions G. Using the relations of orthogonality (8), one finds

$$f(\zeta) = \sum_{h=1}^{\infty} \frac{G_h(\zeta)}{C_h} \int_{-\infty}^{c_2} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot G_h(\tau)\, d\tau + \sum_{h=1}^{\infty} \frac{G_{-h}(\zeta)}{C_{-h}} \cdot \int_{-\infty}^{c_2} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} G_{-h}(\tau) \cdot d\tau \tag{10}$$

where the constants C_h are given by

$$C_h = \int_{-\infty}^{c_2} G_h^2(\tau) \cdot \frac{\tau}{(1-\tau^3)^{7/6}} \cdot d\tau \tag{11}$$

In Equation (10) the limiting process that $c_2 \to 1$ will be carried out. Valid for any value of λ, we have the formula (Appendix 4, Reference 2)

$$\int_{-\infty}^{c_2} G^2(\tau)\frac{\tau}{(1-\tau^3)^{7/6}}\,d\tau = (1-c_2^3)^{5/6}\left(G\frac{\partial^2 G}{\partial\zeta\,\partial\lambda} - \frac{\partial G}{\partial\zeta}\,\frac{\partial G}{\partial\lambda}\right)\Bigg|_{\zeta=c_2}$$

Here the functions G are defined by the differential equation (7) and their development at the point $\zeta = -\infty$ which starts with the term $(-\zeta)^{-3/2}$. The functions G are expressed by hypergeometric series in Equations (18) of Reference 2. For the evaluation of the C_h one needs a representation of the C_h in the vicinity of $\zeta = 1$. According to Equation (18d), Reference 2, one has

$$G^{(a)}(\zeta,\lambda) = \Gamma(\tfrac{1}{2})\left\{\frac{\Gamma(\frac{2}{3}\sqrt{\lambda})\cdot\sin\left(\pi\left(\frac{\sqrt{\lambda}}{3}+\frac{1}{4}\right)\right)}{\Gamma\left(\frac{\sqrt{\lambda}}{3}+\frac{11}{12}\right)\cdot\Gamma\left(\frac{\sqrt{\lambda}}{3}+\frac{7}{12}\right)}(1-\zeta^3)^{\frac{1}{12}-\frac{\sqrt{\lambda}}{3}}\cdot F\left(-\frac{\sqrt{\lambda}}{3}+\frac{1}{12},-\frac{\sqrt{\lambda}}{3}+\frac{7}{12},-\frac{2}{3}\sqrt{\lambda}+1,(1-\zeta^3)\right)\right.$$

$$\left.+\frac{\Gamma(-\frac{2}{3}\sqrt{\lambda})\cdot\sin\left(\pi\left(-\frac{\sqrt{\lambda}}{3}+\frac{1}{4}\right)\right)}{\Gamma\left(-\frac{\sqrt{\lambda}}{3}+\frac{11}{12}\right)\cdot\Gamma\left(-\frac{\sqrt{\lambda}}{3}+\frac{7}{12}\right)}(1-\zeta^3)^{\frac{1}{12}+\frac{\sqrt{\lambda}}{3}}\cdot F\left(\frac{\sqrt{\lambda}}{3}+\frac{1}{12},\frac{\sqrt{\lambda}}{3}+\frac{7}{12},\frac{2}{3}\sqrt{\lambda}+1,(1-\zeta^3)\right)\right\}$$

(13)

If $\zeta \to 1$ the hypergeometric series can be replaced by 1. The determination of the positive eigen values has been shown in Reference 2. The main idea is that the first term in Equation (13) tends, for real positive values of $\sqrt{\lambda}$, to infinity if $\zeta \to 1$. Thus the boundary condition $G = 0$ requires that the coefficient of the first term vanish. Hence

$$\sin\pi\left(\frac{\sqrt{\lambda}}{3}+\frac{1}{4}\right) = 0$$

and

$$\lambda_h = \left(-\tfrac{3}{4} + 3h\right)^2 \qquad h = 1, 2 \cdots \tag{14}$$

Thus the functions G_h will be given for positive values of h as

$$G_h = G^a\left(\zeta, \left(-\tfrac{3}{4} + 3h\right)^2\right)$$

For these values of λ the limiting process that $c_2 \rightarrow 1$ can be carried out in Equation (12) without difficulty. Correcting an error in Reference 2 (Equation (A26)), one finds

$$C_h = \int_{-\infty}^{1} \frac{\zeta}{(1-\zeta^3)^{7/6}} \cdot G^{a\,2}\left(\zeta, \left(-\tfrac{3}{4}+3h\right)^2\right) d\zeta = \frac{\pi \cdot \sqrt{3}}{2\left(-\frac{1}{2}+2h\right)} \cdot \frac{-\frac{1}{6} \cdot \frac{5}{6} \cdots \left(-\frac{1}{6}+(h-1)\right)}{\frac{2}{3} \cdot \frac{5}{3} \cdots \left(\frac{2}{3}+(h-1)\right)} \cdot \frac{\frac{1}{6} \cdot \frac{7}{6} \cdots \left(\frac{1}{6}+(h-1)\right)}{\frac{1}{3} \cdot \frac{4}{3} \cdots \left(\frac{1}{3}+(h-1)\right)}$$

or

$$C_h = -\frac{\pi}{12\left(-\frac{1}{2}+2h\right)} \cdot \frac{\Gamma\left(-\frac{1}{6}+h\right) \cdot \Gamma\left(\frac{1}{6}+h\right)}{\Gamma\left(\frac{2}{3}+h\right) \cdot \Gamma\left(\frac{1}{3}+h\right)} \tag{15}$$

For the negative eigen values one proceeds as follows:

Introducing

$$\tfrac{1}{3} \cdot \sqrt{\lambda_{-h}} = i\nu$$

or

$$\lambda_{-h} = -9\nu^2 \tag{16}$$

and

$$\left(1 - c_2^{\,3}\right) = \varepsilon \tag{17}$$

we write Equation (13) for G in the vicinity of $\zeta = 1$ as

$$G = \Gamma(\tfrac{1}{2}) \left\{ \frac{\Gamma(2i\nu) \cdot \sin \pi(i\nu + \frac{1}{4})}{\Gamma(i\nu + \frac{11}{12}) \cdot \Gamma(i\nu + \frac{7}{12})} (1-\zeta^3)^{\frac{1}{12} - i\nu} + \frac{\Gamma(-2i\nu) \cdot \sin \pi(-i\nu + \frac{1}{4})}{\Gamma(-i\nu + \frac{11}{12}) \cdot \Gamma(-i\nu + \frac{7}{12})} (1-\zeta^3)^{\frac{1}{12} + i\nu} \right\}$$

For brevity we set

$$\Gamma(\tfrac{1}{2}) \frac{\Gamma(2i\nu) \cdot \sin \pi(i\nu + \frac{1}{4})}{\Gamma(i\nu + \frac{11}{12}) \cdot \Gamma(i\nu + \frac{7}{12})} = A(\nu) \cdot e^{i\sigma(\nu)} \tag{18}$$

where $A(\nu)$ and $\sigma(\nu)$ are real. Then,

$$G = 2A(\nu)(1-\zeta^3)^{\frac{1}{12}} \cdot \cos(\sigma(\nu) - \nu \log(1-\zeta^3)) \tag{19}$$

At $\zeta = c_2$ the boundary condition $G = 0$ yields

$$\cos(\sigma(\nu_h) - \nu_h \cdot \log \varepsilon) = 0 \tag{20}$$

Using Equations (17) and (20) and carrying out the limiting process $c_2 \to 1$ as far as practical, one obtains from Equation (19)

$$\frac{\partial G}{\partial \lambda} = \frac{1}{g} \cdot \frac{A}{\nu_h} \cdot \varepsilon^{\frac{1}{12}} \cdot \sin(\sigma(\nu_h) - \nu_h \cdot \log \varepsilon) \cdot (\sigma' - \log \varepsilon)$$

$$\frac{\partial G}{\partial \zeta} = -6A \cdot \varepsilon^{-\frac{11}{12}} \nu_h \cdot \sin(\sigma(\nu_h) - \nu_h \log \varepsilon)$$

Because of Equation (20) the absolute value of the sine function is 1. Thus the values of C_{-h} are given by

$$C_{-h} = -\frac{2}{3} A^2(\nu_h)(\sigma' - \log \varepsilon) \tag{21}$$

C_{-h} tends to positive infinity as $\varepsilon \to 0$. The values of ν_h are determined from Equation (20). One finds

$$\sigma(\nu_h) - \nu_h \log \varepsilon = \pi(\tfrac{1}{2} + h)$$

For small values of ε (c_2 close to 1), the quantity ν will change by only a small amount if h changes by unity. Denoting the change of ν between two consecutive values of h as $\Delta \nu_h$, one thus finds from the last equation

$$\Delta \nu_h (\sigma' - \log \varepsilon) \to \pi \quad \text{as} \quad \varepsilon \to 0$$

With this formula the second expression occurring in Equation (10) can be changed from a summation over h to a summation over the ν_h

$$\sum_{h=1}^{\infty} \frac{G_{-h}(\xi)}{C_{-h}} \cdot \int_{-\infty}^{c_2} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot G_h(\tau)\, d\tau$$

$$= - \sum_{\nu_h} \frac{\Delta \nu_h \cdot \overset{a}{G}(\xi, -9\nu_h^2)}{\pi \cdot \frac{2}{3} \cdot A^2(\nu_h)} \cdot \int_{-\infty}^{c_2} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot \overset{a}{G}(\tau, -9\nu^2)\, d\tau$$

Hence one finds

$$\lim_{c_2 \to 1} \sum_{h=1}^{\infty} \frac{G_{-h}(\xi)}{C_{-h}} \cdot \int_{-\infty}^{c_2} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot G_h(\tau)\, d\tau$$

$$= -\frac{3}{2\pi} \int_0^{\infty} \frac{\overset{a}{G}(\xi, -9\nu^2)}{A^2(\nu)} \left(\int_{-\infty}^{1} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot \overset{a}{G}(\tau, -9\nu^2)\, d\tau \right) \cdot d\nu$$

Thus Equation (10) yields finally

$$f(\xi) = \sum_{n=1}^{\infty} \frac{G^a(\xi, (-\frac{3}{4}+3n)^2}{C_n} \cdot \int_{-\infty}^{1} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot G(\tau, (-\tfrac{3}{4}+3n)^2)\, d\tau$$

$$- \frac{3}{2\pi} \int_{\nu=0}^{\infty} \frac{G^a(\xi, -9\nu^2)}{A^2(\nu)} \left(\int_{-\infty}^{1} f(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot G^a(\tau, -9\nu^2)\, d\tau \right) d\nu \tag{22}$$

SECTION V

THE SOLUTION OF THE BOUNDARY VALUE PROBLEM

By a line ρ = constant = ρ_0 through point A of Figure 3, the region in consideration is divided into an inner portion ($\rho < \rho_0$) and an outer portion ($\rho > \rho_0$). For these regions different representations of the solution containing undetermined coefficients will be chosen. The coefficients will then be found by matching the solutions along the line $\rho = \rho_0$.

In the inner region one has the boundary condition $\psi = 1$ along OA, i.e., for $\xi = -\infty$. A solution of the differential equation (4) which fulfills this boundary condition is obtained for $\lambda = 1/16$. Then ψ is a function of ξ only. It may be denoted by H(ξ). This function fulfills the differential equation

$$\frac{d}{d\xi}\left((1-\xi^3)^{5/6} \cdot H'\right) = 0 \tag{23}$$

and may be defined by the boundary conditions

$$\begin{aligned} H &= 1 \text{ for } \xi \to -\infty \\ H &= 0 \text{ for } \xi = 1 \end{aligned} \tag{23a}$$

The function H(ξ) can be expressed by a linear combination of the symmetric and antisymmetric solution G for $\lambda = 1/16$. The symmetric

solution is simply $G = 1$. Since $G^a(\xi, 1/16)$ is zero for $\lambda = -\infty$, H is expressed by

$$H = 1 + c_3 \cdot G^a(\xi, \tfrac{1}{16})$$

here c_3 is a coefficient which is determined by the condition $H = 0$ at $\xi = 1$. Using Equation (13) one obtains for the value of H at $\xi = 1$ according to the last equation

$$H = 1 + c_3 \cdot \Gamma(\tfrac{1}{2}) \frac{\Gamma(\tfrac{1}{6}) \cdot \sin\frac{\pi}{3}}{\Gamma(1) \cdot \Gamma(\tfrac{2}{3})}$$

Hence

$$c_3 = -\frac{2}{\sqrt{\pi} \cdot \sqrt{3}} \cdot \frac{\Gamma(\tfrac{2}{3})}{\Gamma(\tfrac{1}{6})}$$

and

$$H = 1 - \frac{2}{\sqrt{\pi} \cdot \sqrt{3}} \cdot \frac{\Gamma(\tfrac{2}{3})}{\Gamma(\tfrac{1}{6})} \cdot G(\xi, \tfrac{1}{16}) \tag{24}$$

To the expression H a superposition of the solutions $\psi_{I,h}$ $\psi_{III\,h}$ and $\psi_{IV\,h}$ with undetermined coefficients will be added. Particular solutions $\psi_{I\,h}$ will not occur since they would introduce a singularity at point O. The particular solutions $\psi_{III\,h}$ and $\psi_{IV\,h}$ are also singular at point O, because of the factor $\rho^{-1/2}$ which occurs (see Equation (9)); however, the final solution will not show this singularity. Thus the representation of the inner solution will be given by

$$\psi = \sum_{h=1}^{\infty} a_h \left(\frac{\rho}{\rho_0}\right)^{-\frac{1}{3}+h} \cdot G^a\left(\xi, \left(-\frac{3}{4}+3h\right)^2\right.$$

$$+\left(\frac{\rho}{\rho_0}\right)^{-\frac{1}{12}} \int_0^{\infty} \alpha(\nu) \cdot G^a(\xi, -9\nu^2) \cdot \cos\left(\nu \log \frac{\rho}{\rho_0}\right) d\nu + \left(\frac{\rho}{\rho_0}\right)^{-\frac{1}{12}} \int_0^{\infty} \beta(\nu) \cdot G^a(\xi, -9\nu^2) \sin\left(\nu \log \frac{\rho}{\rho_0}\right) d\nu \tag{25}$$

The constants a_h and the functions $\alpha(\nu)$ and $\beta(\nu)$ will be determined by the matching process along $\rho = \rho_0$.

For the representation of the outer solutions the boundary conditions at the wedge surface BCD must be taken into account. One first constructs a family of particular solutions which fulfills along the line OABCD the boundary condition $\psi = 0$ and which have a singularity at point 0 given by $\psi_{II h}$. These particular solutions have the form

$$\psi_h = \psi_{I h} + \tilde{\psi}_h \tag{26}$$

where the functions $\tilde{\psi}_h$ fulfill the conditions of Tricomi's uniqueness and existence proof (Reference 3). It has been shown in Reference 2 that any solution, which fulfills outside of a line $\rho = \rho_0$ the boundary condition $\psi = 0$ along ABCD, can be represented by a superposition of the expressions (26). Thus the outer solution is represented by

$$\psi = \sum_{h=1}^{\infty} b_h \cdot \rho_0^{\left(-\frac{1}{6}+h\right)} \cdot \psi_h \tag{27}$$

The coefficients b_h will be determined by the matching process. If the choking Mach number is quite close to one, the line $\rho = \rho_0$ lies rather closely to the origin and the singular terms in the expressions (26) will prevail, so that for the purpose of matching along the line $\rho = \rho_0$ the outer solution can be replaced by the expression

$$\psi = \sum_{h=1}^{\infty} \left(\frac{\rho}{\rho_0}\right)^{-(\frac{1}{6}+h)} \cdot G^a\left(\zeta, \left(-\tfrac{3}{4}+3h\right)^2\right)$$

(27a)

By matching the inner and outer representations for the function ψ and their derivatives with respect to ρ along the line $\rho = \rho_0$, one obtains the equations

$$H(\zeta) + \sum_{h=1}^{\infty} a_h \cdot G^a\left(\zeta, \left(-\tfrac{3}{4}+3h\right)^2\right) + \int_0^{\infty} \alpha(\nu) \cdot G^a\left(\zeta, -9\nu^2\right) d\nu$$
$$= \sum_{h=1}^{\infty} b_h \cdot G^a\left(\zeta, \left(-\tfrac{3}{4}+3h\right)^2\right)$$

(28a)

$$\sum_{h=1}^{\infty} a_h \cdot \left(-\tfrac{1}{3}+h\right) \cdot G^a\left(\zeta, \left(-\tfrac{3}{4}+3h\right)^2\right) + \int_0^{\infty} G^a\left(\zeta, -9\nu^2\right)\left(-\tfrac{1}{12}\alpha(\nu) + \nu\beta(\nu)\right) d\nu$$

$$= -\sum_{h=1}^{\infty} \left(-\tfrac{1}{6}+h\right) \cdot b_h \cdot G^a\left(\zeta, \left(-\tfrac{3}{4}+3h\right)^2\right)$$

(28b)

The function H can be developed in terms of the functions G by means of Equation (22). By suitable integration by parts which utilize the differential equation (23) for H and the differential equation (7) for the G and furthermore the boundary conditions for G and H (Equation (23a)) one finds

$$\int_{-\infty}^{1} H(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot G^a(\tau, \lambda_h)\, d\tau = -\frac{3}{2(\lambda_h - \frac{1}{16})}$$

and

$$\int_{-\infty}^{1} H(\tau) \frac{\tau}{(1-\tau^3)^{7/6}} \cdot G^a(\tau, -9\nu^2)\, d\tau = \frac{3}{2(9\nu^2+\frac{1}{16})}$$

With Equations (14) and (16) one thus finds as the representation for H

$$H = -\frac{3}{2} \sum_{h=1}^{\infty} \frac{G^a(\xi, \lambda_h)}{(\lambda_h - \frac{1}{16}) \cdot C_h} - \frac{9}{4\pi} \int_0^{\infty} \frac{G^a(\xi, -9\nu^2)}{(9\nu^2+\frac{1}{16})\, A^2(\nu)} \cdot d\nu$$

Inserting this expression into Equation (27a) one finds by equating the coefficients of the different expressions G

$$a_h - \frac{3}{2} \frac{1}{(\lambda_h - \frac{1}{16}) \cdot C_h} = b_h$$

$$\alpha_\nu = \frac{9}{4\pi} \frac{1}{(9\nu^2+\frac{1}{16}) \cdot A^2(\nu)}$$

Similarly from Equation (27b)

$$a_h \left(-\tfrac{1}{3}+h\right) = -\left(-\tfrac{1}{6}+h\right) \cdot b_h$$

$$-\frac{1}{12}\, \alpha(\nu) + \nu \beta(\nu) = 0$$

Hence

$$a_h = \frac{3}{2} \cdot \frac{1}{(\lambda_h - \frac{1}{16}) \cdot C_h} \cdot \frac{(-\frac{1}{6}+h)}{(-\frac{1}{2}+2h)}\, ; \quad b_h = -\frac{3}{2} \cdot \frac{1}{(\lambda_h - \frac{1}{16}) \cdot C_h} \cdot \frac{(-\frac{1}{3}+h)}{(-\frac{1}{2}+2h)}$$

$$\alpha(\nu) = \frac{9}{4\pi} \cdot \frac{1}{(9\nu^2+\frac{1}{16}) \cdot A^2(\nu)}\, ; \quad \beta(\nu) = \frac{1}{12\nu} \cdot \alpha(\nu)$$

This determines the solution. We are particularly interested in the representation of the inner region, since it gives the pressure distribution along the wind-tunnel wall. It is given by

$$\psi = H(\zeta) + \sum_{h=1}^{\infty} \frac{3}{2} \cdot \frac{1}{(\lambda_h - \frac{1}{6}) \cdot C_h} \cdot \frac{(-\frac{1}{6}+h)}{(-\frac{1}{2}+2h)} \cdot \left(\frac{\rho}{\rho_0}\right)^{-\frac{1}{3}+h} \cdot G^{a}\left(\zeta, \left(-\frac{3}{4}+3h\right)^2\right)$$

$$+ \left(\frac{\rho}{\rho_0}\right)^{-\frac{1}{12}} \cdot \frac{1}{4\pi} \int_0^{\infty} \frac{\left(\cos\left(\nu \log \frac{\rho}{\rho_0}\right) + \frac{1}{12\nu} \sin\left(\nu \log \frac{\rho}{\rho_0}\right)\right)}{\left(\nu^2 + \frac{1}{144}\right) \cdot A^2(\nu)} \cdot G^{a}\left(\zeta, -9\nu^2\right) d\nu \tag{29}$$

SECTION VI

TRANSFORMATION OF EQUATION (29)

A direct evaluation of the last expression would be quite cumbersome because of the occurrence of the complex function $A(\nu)$, defined in Equation (18). One observes, however, that the integrand is an analytic function of ν and therefore by a deformation of the path of integration in the complex ν-plane one can eventually obtain a more convenient expression. Actually, the solution will finally appear as an infinite series which also displays more clearly the behavior of the solution near point O. This transformation will be carried out in the following solutions but instead of the expression ψ we shall determine directly the x coordinate along the wind-tunnel wall as a function of η, for this is the quantity in which we are mainly interested.

In the present simplification of the hodograph equation one has the relation

$$x_\eta = (\gamma+1)^{\frac{1}{3}} \cdot \eta \cdot \psi_\theta$$

for simplicity the stream density at sonic speed (i.e., the flux per unit area) has been put equal to 1. The functions G^a behave along the line $\zeta = -\infty$ as $(-\zeta)^{-3/2}$.

With

$$\frac{\partial \zeta}{\partial \theta} = -\frac{\eta}{\left(\frac{3}{2}\theta\right)^{5/3}} = (-\eta)^{-3/2}(-\zeta)^{\frac{5}{2}}$$

$$\text{and} \quad \frac{\partial \rho}{\partial \theta} = 0 \qquad \text{for } \zeta = -\infty$$

one thus finds from Equation (29), inserting at the same time the expression for H from Equation (24),

$$x_\eta = -\frac{3}{2}(\gamma+1)^{\frac{1}{3}}(-\eta)^{-\frac{1}{2}}\left\{\frac{-2}{\sqrt{\pi}\cdot\sqrt{3}}\,\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})} + \sum_{h=1}^{\infty}\frac{3}{2}\cdot\frac{1}{(\lambda_h-\frac{1}{16})\cdot C_h}\cdot\frac{(-\frac{1}{6}+h)}{(-\frac{1}{2}+2h)}\cdot\left(\frac{\eta}{\eta_0}\right)^{-1+3h}\right.$$

$$\left. + \left(\frac{\eta}{\eta_0}\right)^{-\frac{1}{4}}\cdot\frac{1}{4\pi}\int_0^{\infty}\frac{\cos\left(3\nu\log\frac{\eta}{\eta_0}\right)+\frac{1}{12\nu}\cdot\sin\left(3\nu\log\frac{\eta}{\eta_0}\right)}{\left(\nu^2+\frac{1}{144}\right)\cdot A^2(\nu)}\,d\nu\right\}$$

Here η_0 is the value of η for the point A, η_0 is negative. Then x for the wind-tunnel wall is found by an integration; x = 0 may lie at the intersection of the sonic line with the wall.

$$x = \frac{3}{2}(\gamma+1)^{1/3}(-\eta_0)^{\frac{1}{2}}\left\{-\frac{4}{\sqrt{\pi}\cdot\sqrt{3}}\cdot\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})}\left(\frac{\eta}{\eta_0}\right)^{\frac{1}{2}} + \sum_{h=1}^{\infty}\frac{1}{2}\cdot\frac{1}{(\lambda_h-\frac{1}{16})}\cdot\frac{1}{C_h}\,\frac{1}{(-\frac{1}{2}+2h)}\left(\frac{\eta}{\eta_0}\right)^{-\frac{1}{2}+3h}\right.$$

$$\left. + \left(\frac{\eta}{\eta_0}\right)^{\frac{1}{4}}\cdot\frac{1}{4\pi}\cdot\int_0^{\infty}\frac{\sin\left(3\nu\log\frac{\eta}{\eta_0}\right)}{\left(\nu^2+\frac{1}{144}\right)\cdot A^2(\nu)\cdot 3\nu}\cdot d\nu\right\} \tag{30}$$

The integral which occurs on the right hand side may now be transformed. It may be denoted by J.

$$J = \int_0^\infty \frac{\sin\left(3\nu \log \frac{\eta}{\eta_0}\right)}{\left(\nu^2 + \frac{1}{144}\right) \cdot A^2(\nu) \cdot 3\nu} \, d\nu \tag{31}$$

Inserting into the last expression the value of A from Equation (18), one finds

$$J = \frac{1}{3\pi} \int_0^\infty \frac{\sin\left(3\nu \log \frac{\eta}{\eta_0}\right) \cdot \Gamma\left(i\nu + \frac{11}{12}\right) \cdot \Gamma\left(-i\nu + \frac{11}{12}\right) \cdot \Gamma\left(i\nu + \frac{7}{12}\right) \cdot \Gamma\left(-i\nu + \frac{7}{12}\right)}{\left(\nu^2 + \frac{1}{144}\right) \cdot \nu \cdot \Gamma(2i\nu) \cdot \Gamma(-2i\nu) \cdot \sin\left(\pi\left(i\nu + \frac{1}{4}\right)\right) \cdot \sin\left(\pi\left(-i\nu + \frac{1}{4}\right)\right)} \cdot d\nu$$

Because of the relations

$$\sin\left(\pi\left(i\nu + \tfrac{1}{4}\right)\right) \cdot \sin\left(\pi\left(-i\nu + \tfrac{1}{4}\right)\right) = \tfrac{1}{2} \cos 2\pi i\nu$$

$$\frac{\Gamma\left(i\nu + \frac{11}{12}\right) \cdot \Gamma\left(-i\nu + \frac{11}{12}\right)}{\nu^2 + 1/144} = \Gamma\left(i\nu - \tfrac{1}{12}\right) \cdot \Gamma\left(-i\nu - \tfrac{1}{12}\right)$$

$$\Gamma(2i\nu) \cdot \Gamma(-2i\nu) = \frac{\pi}{-2i\nu \sin(\pi 2i\nu)}$$

this expression can be simplified to

$$J = -\frac{4}{3\pi^2} \cdot i \int_0^\infty tg(\pi 2i\nu) \cdot \Gamma\left(i\nu - \tfrac{1}{12}\right) \cdot \Gamma\left(-i\nu - \tfrac{1}{12}\right) \cdot \Gamma\left(i\nu + \tfrac{7}{12}\right) \cdot \Gamma\left(-i\nu + \tfrac{7}{12}\right) \sin\left(3\nu \log \tfrac{\eta}{\eta_0}\right) d\nu$$

One can write it as

$$J = \frac{4}{3\pi^2} Re \int_0^\infty tg(\pi 2i\nu) \cdot \Gamma\left(i\nu - \tfrac{1}{12}\right) \cdot \Gamma\left(-i\nu - \tfrac{1}{12}\right) \cdot \Gamma\left(i\nu + \tfrac{7}{12}\right) \cdot \Gamma\left(-i\nu + \tfrac{7}{12}\right) e^{3i\nu \log \frac{\eta}{\eta_0}} \, d\nu$$

We set

$$i\nu = u, \text{ then}$$

$$J = \frac{4}{3\pi^2} \cdot Re \cdot \frac{1}{i} \int_0^{i\infty} tg\,2\pi u \cdot \Gamma(u - \tfrac{1}{12}) \cdot \Gamma(-u - \tfrac{1}{12}) \cdot \Gamma(u + \tfrac{7}{12}) \cdot \Gamma(-u + \tfrac{7}{12}) \cdot e^{3u \log \frac{\gamma}{\gamma_0}} du$$

The path of integration is along the positive imaginary u axis. Further simplifications are carried out by means of the formula

$$\Gamma(z) \cdot \Gamma(1 - z) = \frac{\pi}{\sin \pi z}$$

and the following relations for trigonometric functions

$$\cos 2u = 2 \sin(u + \tfrac{\pi}{4}) \cdot \sin(-u + \tfrac{\pi}{4})$$

$$\sin\alpha \cdot \sin(\alpha + \tfrac{\pi}{3}) \cdot \sin(\alpha + \tfrac{2\pi}{3}) = \tfrac{1}{4} \sin 3\alpha$$

Thus one obtains finally

$$J = \frac{8}{3} Re \left[i \int_0^{i\infty} \left[\frac{\sin 2u}{\sin(u + \frac{\pi}{4}) \cdot \sin(3u + \frac{\pi}{4})} \cdot \frac{\Gamma(u - \frac{1}{12}) \cdot \Gamma(u + \frac{7}{12})}{\Gamma(u + \frac{13}{12}) \cdot \Gamma(u + \frac{5}{12})} \right] e^{3u \log \frac{\gamma}{\gamma_0}} du \right.$$

We shall deform the path of integration from the imaginary axis to the positive real axis. Obviously all poles of the integrand lie at the real axis. This deformation will be possible if the contribution to the integral along a line which leads at a far distance from the imaginary axis in the complex u-plane to the real axis, drops out.

By means of the asymptotic expression for the Γ -function which are valid in the entire u-plane with the exception of the negative real axis, one finds that the combined Γ-functions in the bracket of the last expression give a behavior as u^{-1}. The combined trigonometric functions have a periodicity 2 with respect to the real

part of u, with respect to the imaginary part of u it behaves as $e^{-2 \,\mathrm{Im}\, u}$. Since $\frac{\eta}{\eta_0} < 1$ the expression $e^{3u \log \frac{\eta}{\eta_0}}$ decreases exponentially in the positive real u-direction. On the basis of these properties it is not difficult to find a suitable family of contours connecting the real to the imaginary axis and avoiding the poles which lie on the real axis, such that the contribution of the integral tends to zero if u along the contours exceeds any limit. Thus the integration can be carried along the path in the u-plane shown in Figure 5, where the singular points are excluded by small circles. Now because of the factor 1/i in front of the integral the portions of the path of integration along the real axis do not contribute to the real part of the last expression and only the contribution of the circles around the poles is of importance. Thus one finds

$$J = \frac{8}{3} \cdot \pi \cdot \text{Residues of } \frac{\sin 2\pi u \cdot \Gamma(u - \frac{1}{12}) \cdot \Gamma(u + \frac{7}{12})}{\sin(u + \frac{\pi}{4}) \cdot \sin(3u + \frac{\pi}{4}) \cdot \Gamma(u + \frac{13}{12}) \Gamma(u + \frac{5}{12})} \cdot e^{3u \log \frac{\eta}{\eta_0}} \tag{32}$$

The poles of the integrand along the positive real axis lie at

$$u = 1/12, \quad u = -1/4 + h \quad (h = 1, 2 \ldots)$$
$$u = -1/12 + h/3 \quad (h = 1, 2 \ldots)$$

For the points $u = -1/12 + h/3$ one finds by direct evaluation

$$\frac{\sin 2\pi u}{\sin(u + \frac{\pi}{4})} = (-)^{h-1}$$

Thus the residues at the point $u = -1/12 + h/3$ are found to be

$$\frac{-\Gamma(\frac{h}{3} - \frac{1}{6}) \cdot \Gamma(\frac{h}{3} + \frac{1}{2})}{3\pi \, \Gamma(\frac{h}{3} + 1) \cdot \Gamma(\frac{h}{3} + \frac{1}{3})} \cdot \left(\frac{\eta}{\eta_0}\right)^{h - \frac{1}{4}} \qquad h = 1, 2 \ldots$$

For the residues at points u = $-1/4 + h$ one obtains

$$\frac{\Gamma(-\frac{1}{3}+h)\cdot\Gamma(\frac{1}{3}+h)}{\pi\,\Gamma(\frac{5}{6}+h)\cdot\Gamma(\frac{1}{6}+h)}\cdot\left(\frac{\eta}{\eta_0}\right)^{3h-\frac{3}{4}}$$

Finally the residues at the point u = 1/12 is found to be

$$\frac{\Gamma(\frac{2}{3})}{\sqrt{3}\cdot\Gamma(\frac{7}{6})\cdot\Gamma(\frac{1}{2})}$$

Inserting into Equation (30), the Equations (31), (32) and the expressions for the residues just found, introducing furthermore for the λ_h the expressions (14), and for the C_h the expressions (15),

one finds that the residue at the point u = 1/12 cancels the contribution to x caused by the function H (this is the first term of the bracket in Equation (30) and that the residues at the points u = $-1/4 + h$ cancels the contribution of the sum in the bracket of Equation (30). Thus the result assumes the form

$$x = -(y+1)^{\frac{1}{3}}(-\eta_0)^{\frac{1}{2}}\cdot\frac{1}{3\pi}\sum_{h=1}^{\infty}\frac{\Gamma(\frac{h}{3}-\frac{1}{6})\cdot\Gamma(\frac{h}{3}+\frac{1}{2})}{\Gamma(\frac{h}{3}+1)\cdot\Gamma(\frac{h}{3}+\frac{1}{3})}\cdot\left(\frac{\eta}{\eta_0}\right)^{h} \tag{33}$$

SECTION VII

DISCUSSION

The expression for ψ has been found with the assumption that $\psi = 1$ along the wind-tunnel wall. For a wind tunnel of height H, the result must be multiplied with H/2. η_0 is the value of η at point A and point A corresponds to the choking Mach number. By the definition of η, Equation (1) and the relation

$$M - 1 = \frac{\gamma + 1}{2}\left(\frac{w}{w^*} - 1\right)$$

one finds

$$M - 1 = \frac{1}{2}(\gamma + 1)^{2/3} \cdot \eta$$

Thus Equation (33) can be written as

$$\frac{x}{H} = \frac{\sqrt{2}}{6\pi}\left(1 - M_{choking}\right)^{\frac{1}{2}} \cdot E\left(\frac{1 - M}{1 - M_{choking}}\right)$$

where

$$E\left(\frac{1 - M}{1 - M_{choking}}\right) = -\sum_{h=1}^{\infty} \frac{\Gamma\left(\frac{h}{3} - \frac{1}{6}\right) \cdot \Gamma\left(\frac{h}{3} + \frac{1}{2}\right)}{\Gamma\left(\frac{h}{3} + 1\right) \cdot \Gamma\left(\frac{h}{3} + \frac{1}{3}\right)} \left(\frac{1 - M}{1 - M_{choking}}\right)^{h}$$

The function E is shown in Figure 6. One will surmise that the continuation of the pressure distribution into the supersonic region is the analytical continuation of the above expression. To show this, one must follow the solution through the supersonic region of the hodograph. One will remember that the flow in the vicinity of the centerline of a Laval nozzle maps into three sheets of the hodograph, and that the flow at the wind-tunnel wall is equivalent to such a flow.

In the transonic approximation it is possible to interpret the expression $\frac{1 - M}{1 - M_{choking}}$ as $\frac{p - p^*}{p_{choking} - p^*}$ where p is the pressure

p^* is the pressure at sonic velocity. The investigations have been carried out for a wedge in a wind tunnel, but they have a more general validity. The shape of the body is expressed by the functions

which occur in Equation (26). However, by the assumption of a very wide tunnel this function is eliminated from the future computations. So the result obtained in Equation (33) is valid for any body shape, if the wind tunnel is sufficiently wide.

In this representation the governing parameters are the wind-tunnel width and the choking Mach number. In the paper of Marschner, Reference 1, a connection has been established between the choking Mach number and the dimensions of a diamond shaped airfoil

$$1 - M_{choking} = 1.127 \left\{\frac{L}{H}\right\}^{2/5} \cdot \theta_o^{8/15}$$

Thus the wall-pressure distribution can be connected to the size of a diamond profile.

In Marschner's paper a criterion for the validity of the approximation has been derived. It is given by the fact that the value of ρ_o must be small in comparison to the value of ρ at the point C, i.e., at the sonic point of the shoulder characteristic. This criterion requires that

$$\left(\frac{L}{H}\right)^{6/5} \cdot \theta_o^{-2/5} \ll 1$$

Thus the accuracy of the results depends not only upon the choking Mach number but also upon the slenderness of the profile. Also from a mathematical point of view the result offers some interest. In contrast to Equation (29), only integral powers of η/η_o occur in Equation (33). Thus the singularities at point 0 introduced in the individual particular solutions $\psi_{III h}$ and $\psi_{IV h}$ cancel out by the superposition in the Fourier integral.

If one has to investigate the vicinity of point 0 one will ask which particular solutions of the form (5) would be suitable to represent the solution in the vicinity of point 0 if one considers the boundary condition imposed by the nature of the problem. (In this

regard the conditions imposed in the formulation of the eigen value problem are quite artificial.) These conditions require that the function is antisymmetric with respect to AO (except for a solution $\psi = 1$) and furthermore we know that no singularity propagates along the limiting characteristic DO. Finally point O must not map to infinity. By the condition $\psi = 0$ along OA we are restricted, as previously, to solutions $G^{(a)}$. Their behavior along the limiting characteristic DO can be investigated by writing the arising expressions ψ in the following form which can be obtained by a combination of Equations (5) and (13).

$$\psi = \rho^{-\frac{1}{12}+\frac{\sqrt{\lambda}}{3}} \cdot G^{a}(\zeta,\lambda) = \eta^{-\frac{1}{4}+\sqrt{\lambda}} \cdot \Gamma(\tfrac{1}{2}) \left\{ \frac{\Gamma(\frac{2}{3}\sqrt{\lambda}) \cdot \sin(\pi(\frac{\sqrt{\lambda}}{3}+\frac{1}{4}))}{\Gamma(\frac{\sqrt{\lambda}}{3}+\frac{11}{12}) \cdot \Gamma(\frac{\sqrt{\lambda}}{3}+\frac{7}{12})} + \frac{\Gamma(-\frac{2}{3}\sqrt{\lambda}) \cdot \sin(\pi(-\frac{\sqrt{\lambda}}{3}+\frac{1}{4}))}{\Gamma(-\frac{\sqrt{\lambda}}{3}+\frac{11}{12}) \cdot \Gamma(-\frac{\sqrt{\lambda}}{3}+\frac{7}{12})} \cdot (1-\zeta^3)^{\frac{2}{3}\sqrt{\lambda}} \right\}$$

The hypergeometric functions which occur in Equation (13) have been replaced by 1 since ζ is close to 1. The last equation shows that the limiting Mach wave would map to infinity (for $\sqrt{\lambda} < 0$), or a singularity would propagate along it (for $\sqrt{\lambda} > 0$) if the second term in the bracket is different from zero. From the condition that the coefficient of the second term vanishes one obtains certain values of λ (incidentally, they are different from the values of λ for the eigen functions). If one determines for the particular solutions determined with these values of λ, the expressions for x along the line OA one finds the powers of η/η_0 encountered in Equation (33). So the form of Equation (33) is not unexpected. Nevertheless, it does not seem practical to introduce the particular solution which appears in Equation (33) right at the beginning for then it would have been very difficult to carry out the matching process along the line $\rho = \rho_0$.

From the form (33) one can naturally obtain a representation for ψ along the limiting Mach wave DO. It proceeds also in power of η/η_0 and shows that ψ is different from zero along the limiting Mach wave. Thus the artificial boundary condition $\psi = 0$ along $\zeta = c_2$ will not cause the solution ψ to be zero along DO after the limiting process $c_2 \rightarrow 1$ has been carried out.

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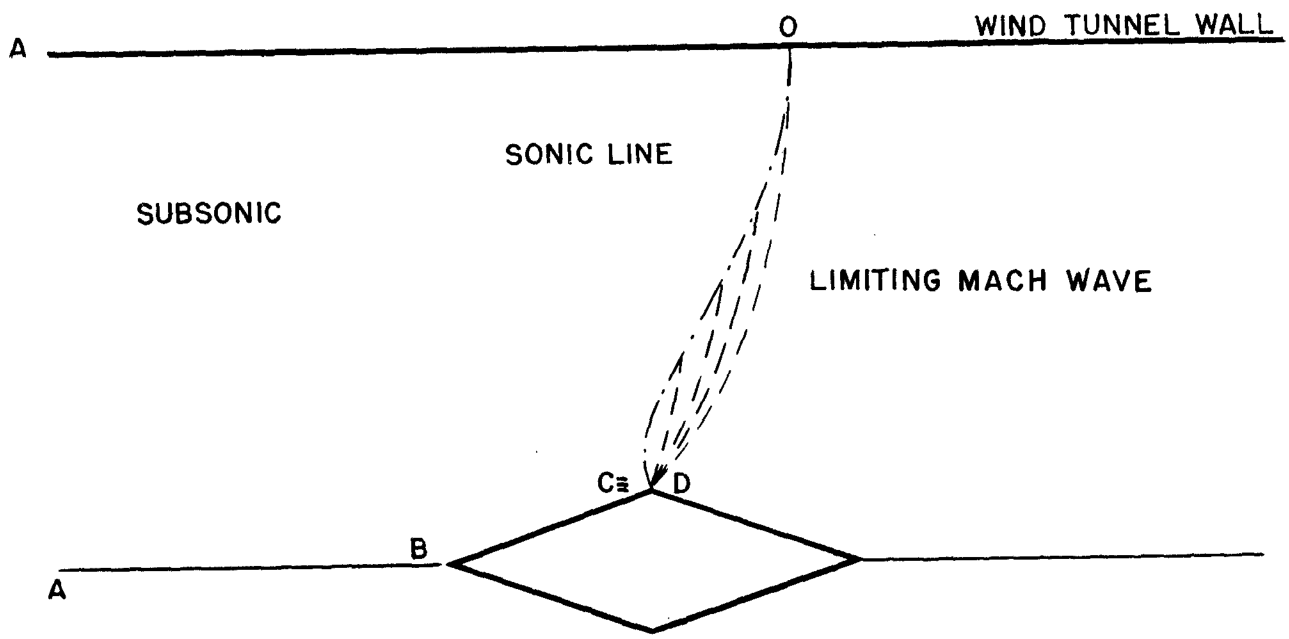

Figure 1: Diamond Profile in a Choked Wind Tunnel

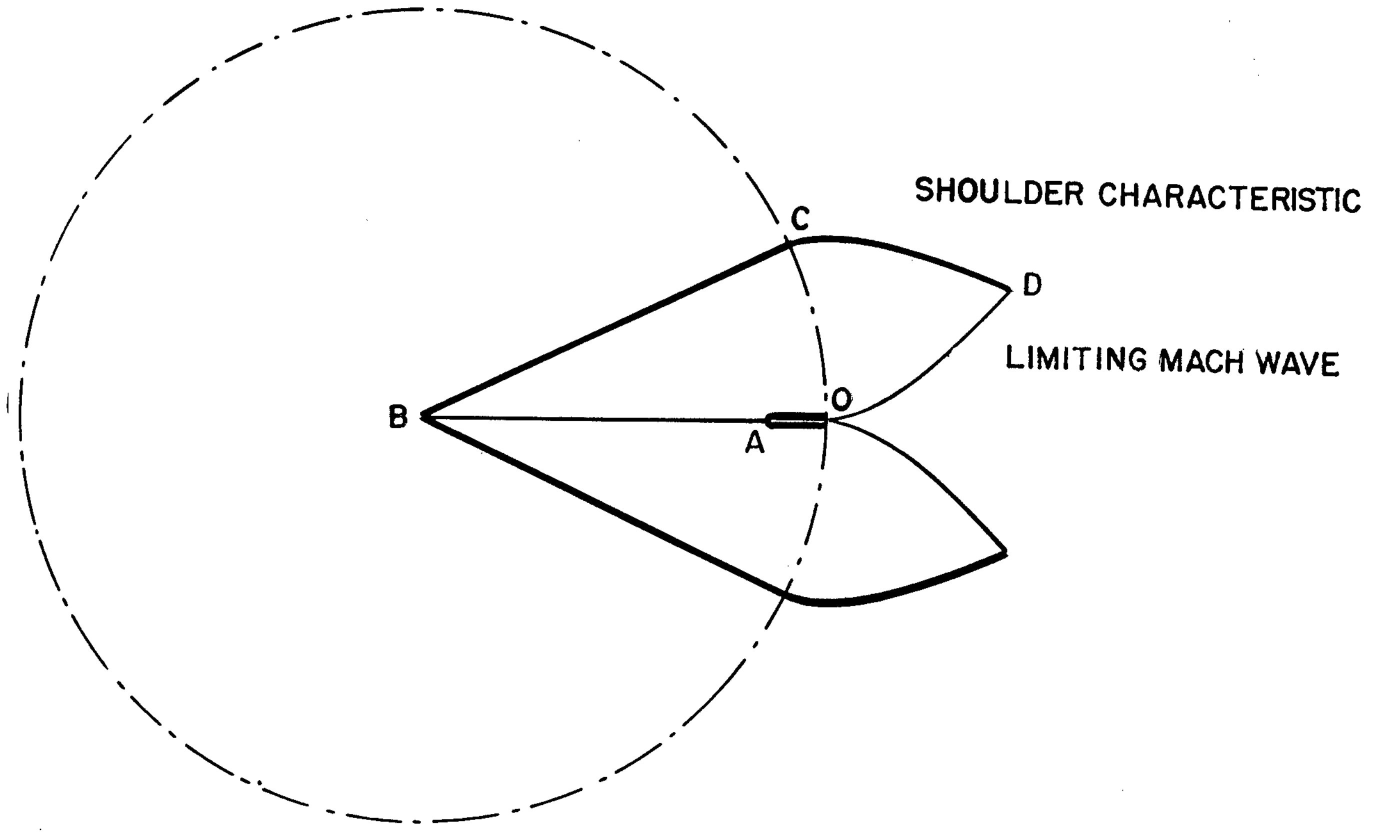

Figure 2: Hodograph Map of Figure 1

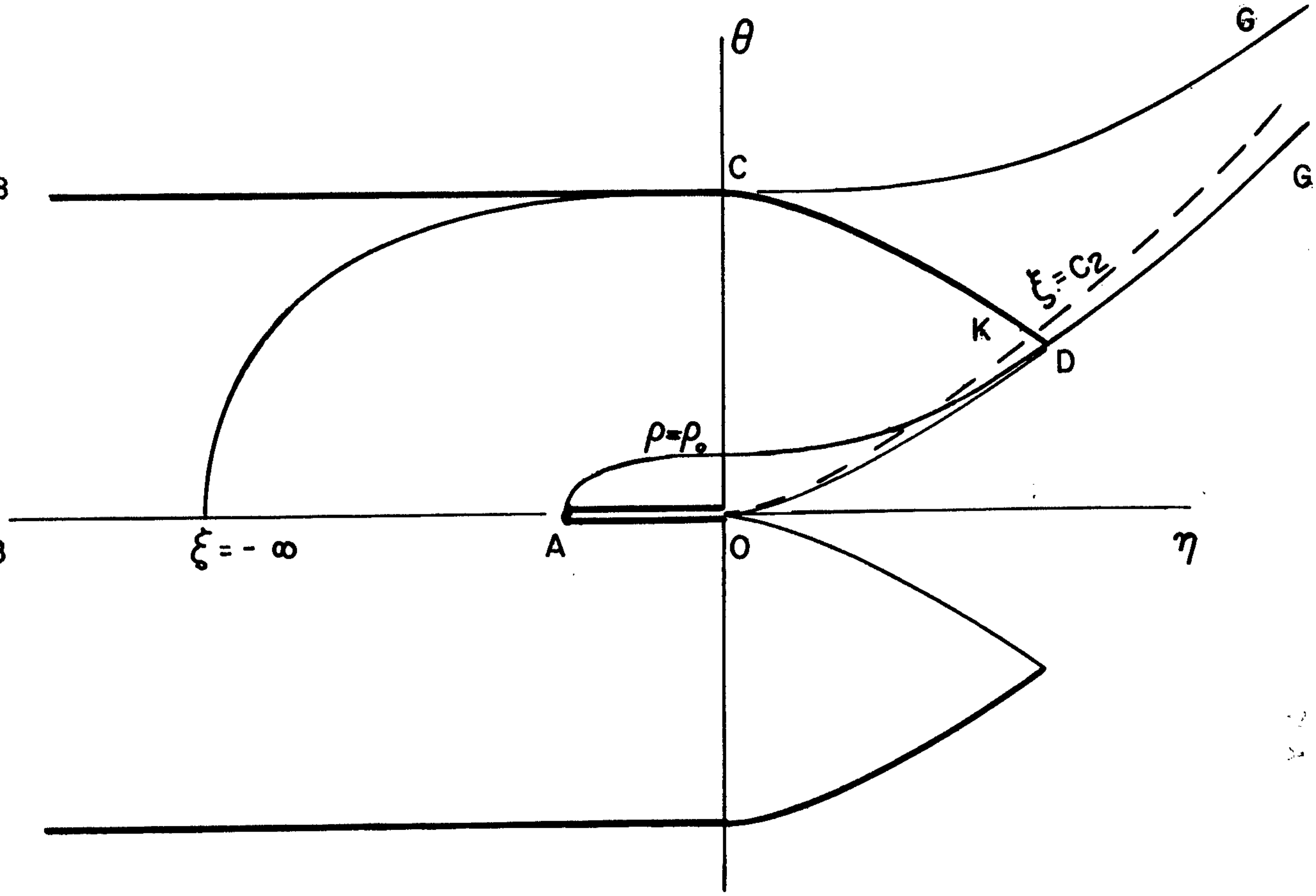

Figure 3: Map of Figure 1 in the η , θ -Plane

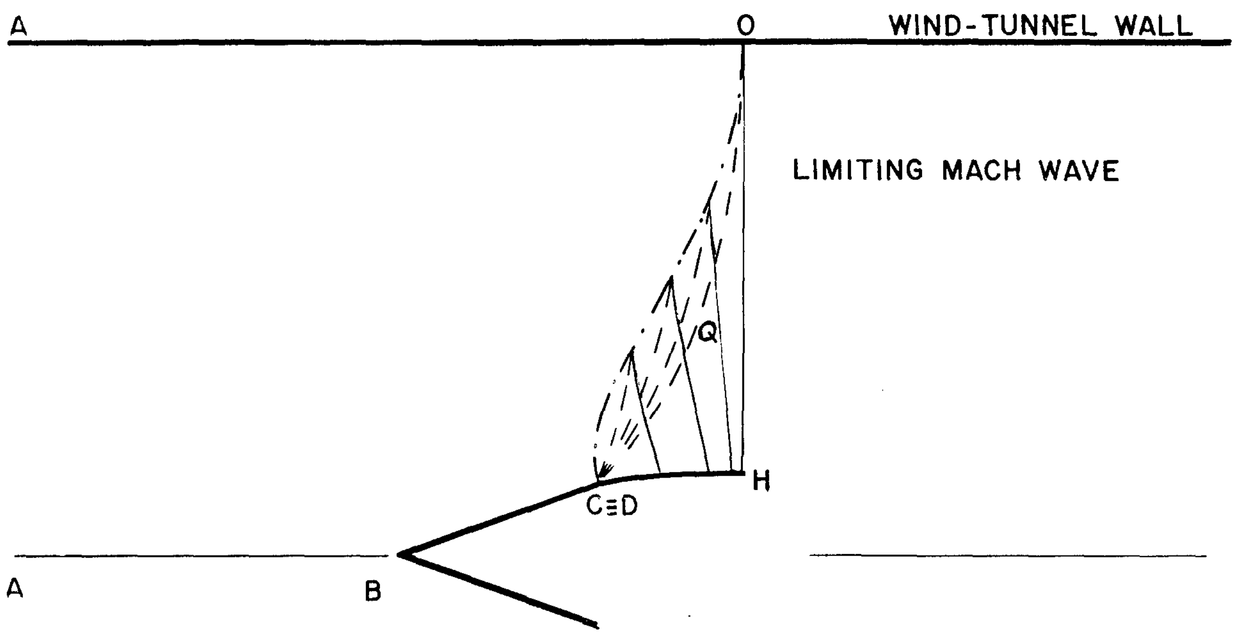

Figure 4: Body With Rear Part Which Does Not Reflect Waves

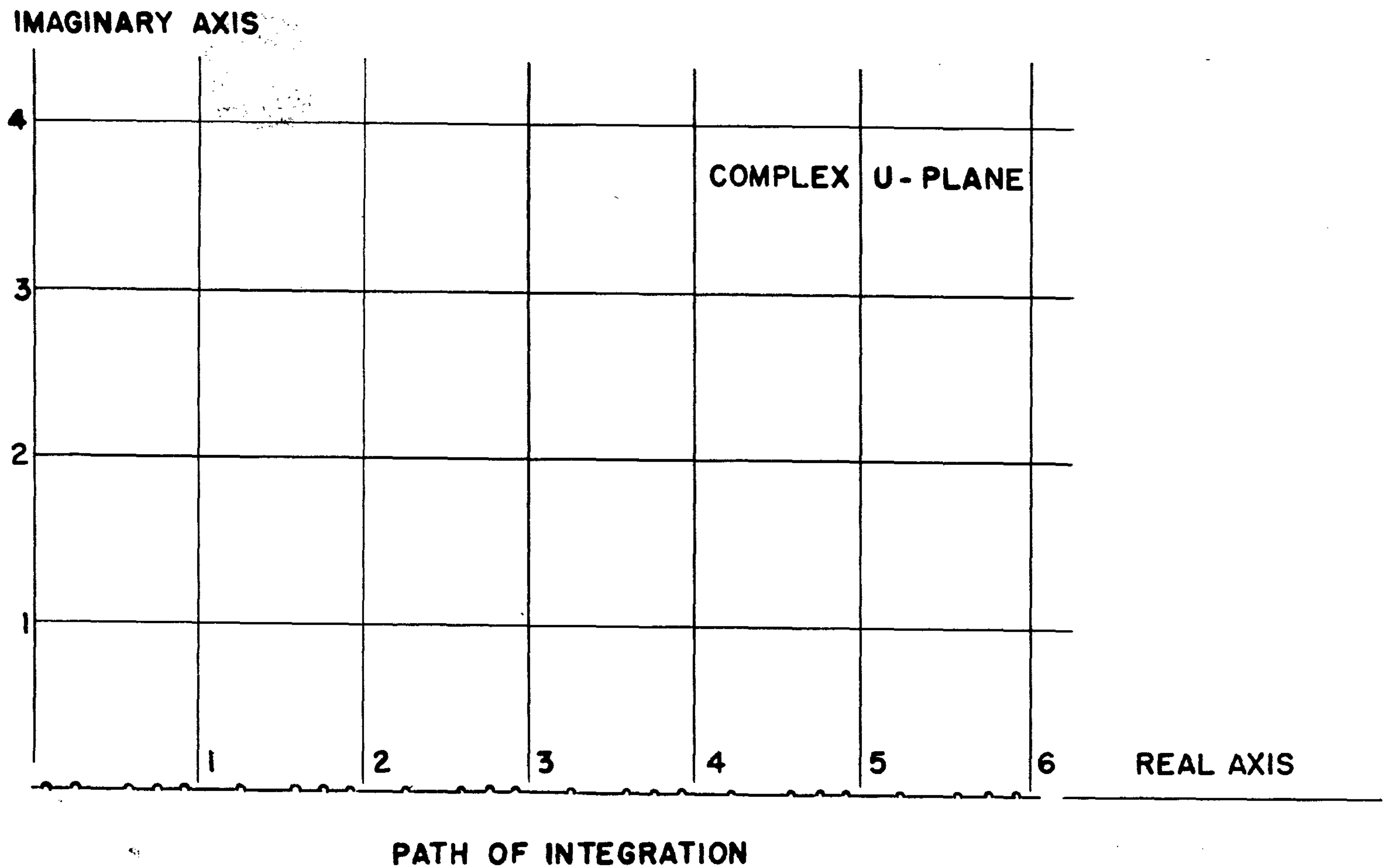

Figure 5: Path of Integral in Complex u-Plane

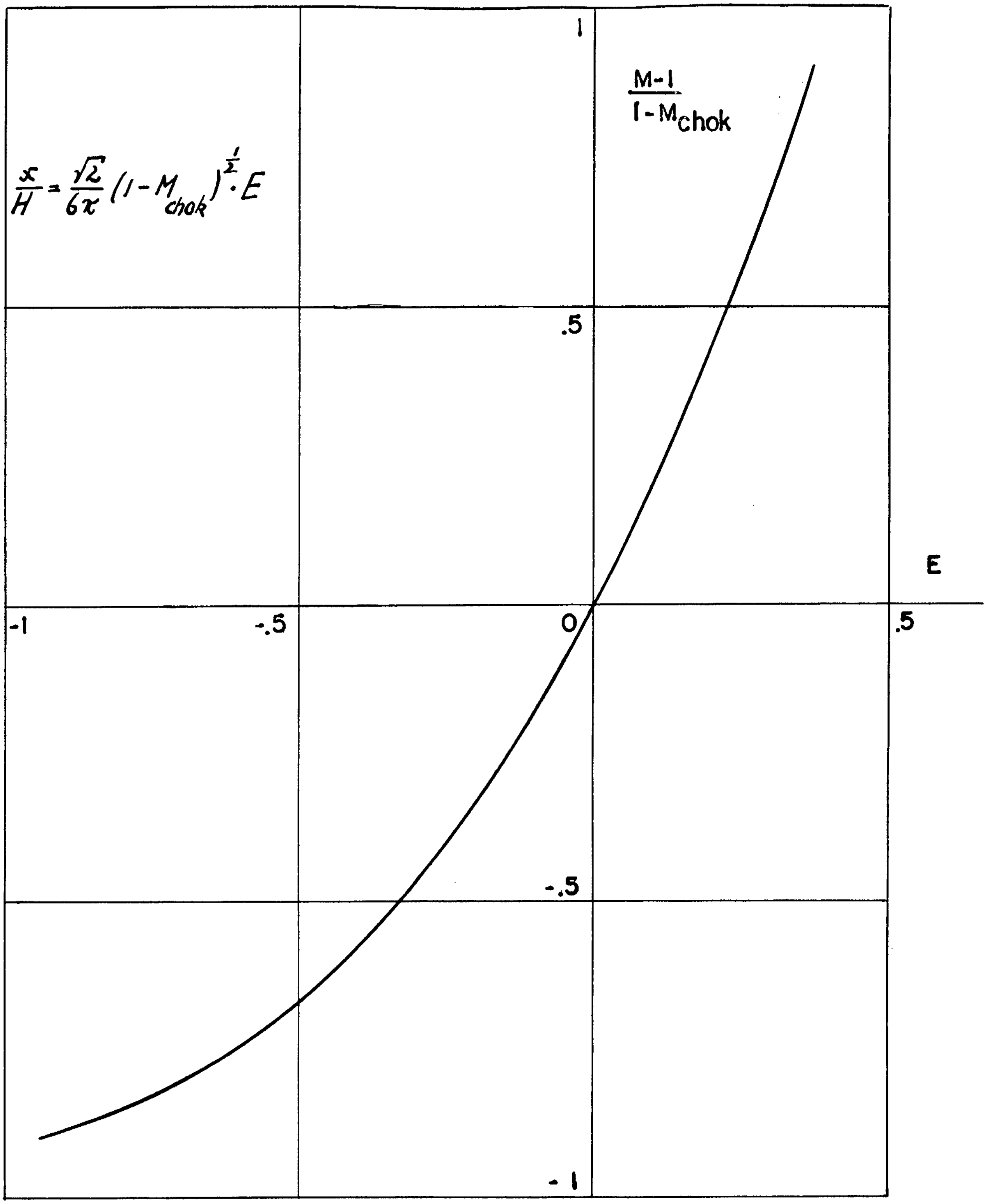

Figure 6: Mach Number Distribution Along Wind Tunnel Wall